WHITE

NAME: Solutions **STUDENT #:**

- There is a total of 42 marks; the maximum grade is 40 (2 bonus marks)
- Check that you have a total of 6 distinct pages and notify your TA if this is not the case
- Calculators are not allowed
- Phones and other devices should be turned off and hidden
- Have your student card face up on your desk
- You have to show all your work for all the questions, except the True/False and multiple choice

Question 1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 8 & 1 & 0 \\ 4 & 0 & 2 \end{bmatrix}$. Find $\det(A)$ *using the cofactor expansion method.* [5 marks]

$$\det(A) = (-1)^{1+1} \cdot 1 \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} + (-1)^{1+2} 2 \begin{vmatrix} 8 & 0 \\ 4 & 2 \end{vmatrix} + (-1)^{1+3} 3 \begin{vmatrix} 8 & 1 \\ 4 & 0 \end{vmatrix}$$

$$= 2 - 2 \cdot 16 + 3(-4) = -42$$

Question 2. Consider the system

$$\begin{aligned} 2x_1 + x_3 + x_4 &= 2 \\ x_2 - x_3 + x_4 &= -1 \\ 4x_1 - x_2 + 3x_3 + x_4 &= 5 \end{aligned}$$

a. Write the augmented matrix of the system. [2 marks]

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 4 & -1 & 3 & 1 & 5 \end{array}\right]$$

b. Row reduce the augmented matrix to **R**REF. [4 marks]

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 4 & -1 & 3 & 1 & 5 \end{array}\right] \xrightarrow{R_3' = R_3 - 2R_1} \left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & -1 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3' = R_3 - R_2} \left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1' = \frac{1}{2}R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1/2 & 1/2 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

c. Which variables are basic and which are free? Tick (✓) the correct box. [2 marks]

Variable	Basic	Free
x_1	✓	
x_2	✓	
x_3		✓
x_4		✓

d. This system has infinitely many solutions. Use the RREF to write the solutions of the system in vector parametric form. [4 marks]

The system corresponding to the RREF is

$$x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 = 1 \implies x_1 = 1 - \frac{1}{2}x_3 - \frac{1}{2}x_4$$

$$x_2 - x_3 + x_4 = -1 \implies x_2 = -1 + x_3 - x_4$$

We set $x_3 = s$, $x_4 = t$ and we have

$$\begin{array}{llll} & \text{constant} & s & t \\ x_1 = & 1 & -\frac{1}{2}s & -\frac{1}{2}t \\ x_2 = & -1 & s & -t \\ x_3 = & & s & \\ x_4 = & & & t \end{array}$$

So the solutions to the system are given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1/2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1/2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad s,t \in \mathbb{R}$$

Question 3. Find the inverse of $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, if it exists. If the inverse does not exist, explain why. [5 marks]

$\left.\begin{array}{l}\text{We have that } \det(A) = 1\cdot 3\cdot 1 = 3 \neq 0, \\ \text{so } A \text{ is invertible.}\end{array}\right\}$ optional

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2' = \frac{R_2}{3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2' = R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/3 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1' = R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1/3 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

We conclude that

$$A^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1/3 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 4. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 2 \\ -1 & -3 & 3 \end{bmatrix}$. You are given that A can be row reduced to $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ by applying the following operations (in that order):

1) $R_3' = R_3 + R_2$
2) $R_2' = R_2 - R_1$
3) $R_3' = \frac{1}{5} R_3$.

a. Find the corresponding elementary matrix for each of the above operations. [3 marks]

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

b. Find the inverse of each of the elementary matrices in a. [3 marks]

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

c. Use the above to give an LU-factorization of A. (You can give L as a product of matrices without carrying out the calculations.) [4 marks]

We have that $A = LU$ with

$$U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$L = E_1^{-1} E_2^{-1} E_3^{-1}$$

3×2 2×2

Question 5. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 3 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 \\ -2 & 1 \end{bmatrix}$. State which of the products AB or BA is not defined and compute the other. [4 marks]

BA is not defined.

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 3 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -6 & 3 \\ 13 & 6 \end{bmatrix}$$

True/False and multiple choice questions

Question 6. Circle T for *True* and F for *False*. Do not justify your answers, just circle your choice. [1 mark *each*]

1. The REF of a matrix is unique. **T** **F**
2. If both A and B are $n \times n$ matrices then $AB = BA$. **T** **F**
3. Let B be the matrix obtained by performing the row operation $R_2' = R_2 + 5R3$ on a square matrix A. Then $\det(B) = 5\det(A)$. **T** **F**
4. A system whose coefficient matrix is invertible has *always* a unique solution. **T** **F**

Question 7. Consider the system with augmented matrix

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & k & h \end{array}\right],$$

where $h, k \in \mathbb{R}$. For which h and k does the system have a unique solution? Circle the correct answer. Do not justify your answer, just circle one letter. [2 marks]

A $k = 2, h = 0$ **B** $k = 0, h \neq 0$ **C** There are no such k, h

2×3 2×2

Question 5. Let $A = \begin{bmatrix} 0 & 0 & 5 \\ 1 & 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 0 & 5 \end{bmatrix}$. State which of the products AB or BA is not defined and compute the other. [4 marks]

AB is not defined

$$BA = \begin{bmatrix} -2 & 1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 5 \\ 1 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -4 \\ 5 & 15 & 30 \end{bmatrix}$$

True/False and multiple choice questions

Question 6. Circle T for *True* and F for *False*. Do not justify your answers, just circle your choice. [1 mark *each*]

1. Let B be the matrix obtained by performing the row operation $R_2' \leftrightarrow R2$ on a square matrix A. Then $\det(B) = \det(A)$. **T** **F**

2. The RREF of a matrix is unique. **T** **F**

3. If both A and B are $n \times n$ matrices then $\det(A + B) = \det(A) + \det(B)$. **T** **F**

4. A system whose coefficient matrix is invertible has no solutions. **T** **F**

Question 7. Consider the system with augmented matrix

$$A = \left[\begin{array}{ccc|c} -2 & 2 & 4 & 3 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & k & h \end{array}\right],$$

where $h, k \in \mathbb{R}$. For which h and k does the system have a unique solution? Circle the correct answer. Do not justify your answer, just circle one letter. [2 marks]

A There are no such k, h **B** $k = 0, h \neq 0$ **C** $k = 4, h = 2$